1.0

4.5
5.0
5.6

2.8　2.5

3.2　2.2

3.6

4.0　2.0

1.1

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1.25　1.4　1.6

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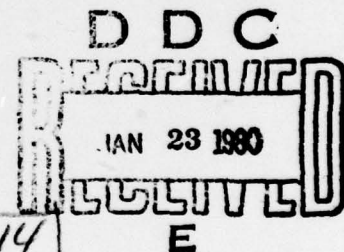
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SOME CONTINUITY PROPERTIES OF POLYHEDRAL MULTIFUNCTIONS.

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SOME CONTINUITY PROPERTIES
OF POLYHEDRAL MULTIFUNCTIONS

Stephen M. Robinson

ABSTRACT

A multifunction is _polyhedral_ if its graph is the union of
finitely many polyhedral convex sets. This paper points out some
fairly strong continuity properties that such multifunctions
satisfy, and it shows how these may be applied to such areas as
linear complementarity and parametric programming.

SIGNIFICANCE AND EXPLANATION

In analyzing nonlinear programming problems and more general equilibrium problems, we sometimes use as tools certain functions, called "multifunctions," whose values are sets instead of points. In general these multifunctions cannot be expected to have good continuity properties. However, we show here that if their graphs have a special structure---which is found in the cases cited above-- then they do obey unexpectedly strong continuity conditions. Some applications are pointed out.

SOME CONTINUITY PROPERTIES
OF POLYHEDRAL MULTIFUNCTIONS

Stephen M. Robinson

1. Introduction. This paper deals with some useful continuity properties of a class of
multivalued functions which we call polyhedral. We begin here by recalling some basic
properties of multifunctions, then in Section 2 we prove the main results, and in Section
3 we show how these may be applied to problems in parametric programming. We also point
out (in Section 2) applications of our results to linear complementarity problems.

Multivalued functions, or multifunctions, are functions whose values are sets instead
of points. Given such a multifunction, say F, from $\mathbb{R}^n$ to $\mathbb{R}^m$, we define the graph
of F to be the set

$$\Gamma_F := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x)\} \ .$$

The effective domain of F (dom F) is $\pi_1(\Gamma_F)$, and the image of F (im F) is $\pi_2(\Gamma_F)$,
where π_1 and π_2 are the canonical projections from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ and $\mathbb{R}^m$
respectively. The multifunction F is called closed or convex if Γ_F is closed or
convex. The inverse of F is the multifunction from $\mathbb{R}^m$ to $\mathbb{R}^n$ whose graph is
$\{(y,x) \mid (x,y) \in \Gamma_F\}$.

Our concern here will be with polyhedral multifunctions: i.e., those whose graphs
are unions of finitely many polyhedral convex sets, called components. This class of
multifunctions can be shown to be closed under (finite) addition, scalar multiplication,
and (finite) composition. An especially simple example of a polyhedral multifunction
is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$: its graph has just one component, a sub-
space of $\mathbb{R}^n \times \mathbb{R}^m$. Of course, this multifunction is also convex. Another example of
a polyhedral convex multifunction is the solution set of a system of linear inequalities
and equations, regarded as a function of the right-hand side: let C be a nonempty
polyhedral convex set in $\mathbb{R}^n$ and K be a nonempty polyhedral convex cone in $\mathbb{R}^m$, and
let A be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. For each $x \in \mathbb{R}^n$ define $F(x)$ by

$$F(x) := \begin{cases} Ax-K , & x \in C \\ \phi , & x \notin C . \end{cases}$$

Then for $b \in \mathbb{R}^m$ one has

$$F^{-1}(b) = \{x \in C \mid Ax-b \in K\},$$

so F^{-1} displays the solution set as a function of b . Evidently F and F^{-1} are polyhedral convex multifunctions.

Examples of important nonconvex polyhedral multifunctions are provided by linear generalized equations [4]. To illustrate these, let C be a nonempty polyhedral convex set, and let ψ_C be the indicator function of C :

$$\psi_C(x) := \begin{cases} 0 , & x \in C \\ +\infty , & x \notin C . \end{cases}$$

The subdifferential of ψ_C then yields the (outward) normal cone to C :

$$\partial \psi_C(x) = \begin{cases} \{y \in \mathbb{R}^n \mid \forall c \in C , \langle y, c-x \rangle \leq 0 \}, & x \in C \\ \phi , & x \notin C ; \end{cases}$$

see [6] for details. If A is a linear transformation from $\mathbb{R}^n$ to itself, and if $a \in \mathbb{R}^n$, we may study the linear generalized equation

$$0 \in A x - a + \partial \psi_C(x) . \tag{1}$$

If $C = \mathbb{R}^n$ this just says $Ax - a = 0$, whereas if $C = \mathbb{R}^n_+$ (the non-negative orthant) it is equivalent to

$$Ax - a \geq 0 , \quad x \geq 0 , \quad \langle x, Ax - a \rangle = 0 ;$$

i.e., (1) formulates the linear complementarity problem. By taking various choices of C one may also cast problems of linear or quadratic programming into the form (1). An example, to which we shall return in Section 3, is provided by letting P and Q be nonempty polyhedral convex cones in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, and

$$A := \begin{bmatrix} H & -D^T \\ D & 0 \end{bmatrix}, \quad a := \begin{bmatrix} -c \\ b \end{bmatrix} , \quad x := \begin{bmatrix} y \\ u \end{bmatrix} , \quad C := P \times Q ,$$

where D is a $q \times p$ matrix, H is a symmetric $p \times p$ matrix, $c, y \in \mathbb{R}^p$ and $b, u \in \mathbb{R}^q$. Then (1) is equivalent to

$$Hy - D^Tu + c \in P^* , \qquad Dy - b \in Q^*$$
$$y \in P , \qquad u \in Q , \qquad\qquad (2)$$
$$\langle y, Hy - D^Tu + c \rangle = 0 , \qquad \langle u, Dy - b \rangle = 0 ,$$

which are the necessary optimality conditions for the quadratic programming problem

$$\text{minimize} \quad \tfrac{1}{2} \langle y, Hy \rangle + \langle c, y \rangle$$
$$\text{subject to} \quad Dy - b \in Q^* \qquad\qquad (3)$$
$$y \in P .$$

Here Q^* denotes the _dual cone_ of Q :

$$Q^* := \{ w \in \mathbb{R}^q \mid \forall q \in Q , \langle w, q \rangle \geqq 0 \},$$

and similarly for P^* . By appropriate choice of P and Q , we can write any quadratic programming problem in the form (3). Thus the general form (1) has a wide range of practical applications.

If we now define, for $x \in \mathbb{R}^n$,

$$F(x) := Ax + \partial \psi_C(x) , \qquad\qquad (4)$$

then for any $a \in \mathbb{R}^n$ one has

$$F^{-1}(a) = \{ x \mid 0 \in Ax - a + \partial \psi_C(x) \} :$$

i.e., the solution set of (1). However, it is easy to show that the graph of $\partial \psi_C$ is a finite union of polyhedral convex sets , so that F (as the sum of $\partial \psi_C(\cdot)$ and $A(\cdot)$) is polyhedral, and so therefore is F^{-1} .

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2. _Continuity results_. Polyhedral multifunctions, whether convex or not, have some strong continuity properties not shared by multifunctions in general, and in this section we shall discuss some of these. From now on, the symbol $\|\cdot\|$ will stand for the Euclidean norm unless otherwise stated, and we shall use B to denote the Euclidean unit ball:

$$B := \{x \mid \|x\| \leq 1\}.$$

The first continuity property is an extension of the Lipschitz condition to multi-functions. Given a multifunction $F: \mathbb{R}^n \to \mathbb{R}^M$, we say that F is <u>locally upper Lipschitzian</u> at a point x_0 <u>with modulus</u> λ , if for some neighborhood N of x_0 and all $x \in N$, $F(x) \subset F(x_0) + \lambda\|x-x_0\|B$. Note that this implies either $x_0 \in$ dom F or $x_0 \notin$ cl dom F .

It is clear that not all multifunctions can be expected to satisfy this property. What we shall show is that polyhedral multifunctions do satisfy it, and moreover that the Lipschitz constant does not depend on the point x_0 . We begin with a lemma.

LEMMA: <u>Let</u> $P : \mathbb{R}^n \to \mathbb{R}^m$ <u>be a nonempty polyhedral multifunction and let</u> G_i , $1 \leq i \leq k$, <u>be its components. Let</u> $x_0 \in$ dom P, <u>and define</u> $I := \{i \mid x_0 \in \pi_1(G_i)\}$, <u>where</u> π_1 <u>is the canonical projection of</u> $\mathbb{R}^n \times \mathbb{R}^m$ <u>on</u> $\mathbb{R}^n$. <u>Then there exists a neighborhood</u> $U(x_0)$ <u>such that</u>

$$(U \times \mathbb{R}^m) \cap \text{graph } P \subset \bigcup_{i \in I} G_i .$$

PROOF: For each i , both $\{x_0\} \times \mathbb{R}^m$ and G_i are nonempty polyhedral convex sets in $\mathbb{R}^{n+m}$. If $i \notin I$ they do not meet, and thus by [6, Cor. 19.3.3] they can be strongly separated. In particular, then, there is some neighborhood $U_i(x_0)$ such that $(U_i \times \mathbb{R}^m) \cap G_i = \emptyset$. Let $U := \bigcap_{i \notin I} U_i$; this U is a neighborhood of x_0 because the number of components is finite. Clearly,

$$(U \times \mathbb{R}^m) \cap \text{graph } P \subset (\bigcup_{i=1}^{k} G_i) \setminus (\bigcup_{i \notin I} G_i) \subset \bigcup_{i \in I} G_i ,$$

and this completes the proof.

This lemma tells us that if $x \in U$ and $y \in P(x)$, then (x,y) belongs to a component G_i which also contains (x_0,y_0) for some $y_0 \in P(x_0)$. That fact, together with known results about polyhedral convex sets, will suffice to prove our first continuity result.

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PROPOSITION 1: Let F be a polyhedral multifunction from $\mathbb{R}^n$ into $\mathbb{R}^m$. Then there exists a constant λ such that F is locally U. L.(λ) at each $x_0 \in \mathbb{R}^n$.

PROOF: If graph F is empty there is nothing to prove. Otherwise let G_i, $1 \le i \le k$, be the components of F, and for each i let a multifunction F_i be defined by

$$F_i(w) = \{z \mid (w,z) \in G_i\} = \pi_2[G_i \cap \pi_1^{-1}(w)] \ .$$

By a theorem of Walkup and Wets [7], the intersection $G_i \cap \pi_1^{-1}(\cdot)$ is Lipschitzian in the Hausdorff metric whenever it is nonempty. Thus there is some λ_i such that if $F_i(w_1) \ne \phi \ne F_i(w_0)$ then $F_i(w_1) \subset F_i(w_0) + \lambda_i \|w_1 - w_0\|B$. Let $\lambda := \max_{i=1}^{k} \lambda_i$, and choose any $x_0 \in \mathbb{R}^n$. If $x_0 \notin \text{dom } F$ then since dom F is a finite union of polyhedral convex sets, some neighborhood of x_0 is disjoint from dom F; hence F is U.L.(λ) at x_0 with respect to that neighborhood. If $x_0 \in \text{dom } F$, define $I := \{i \mid x_0 \in \pi_1(G_i)\}$; by Lemma 1 there exists a neighborhood $U(x_0)$ such that

$$(U \times \mathbb{R}^m) \cap \text{graph } F \subset \bigcup_{i \in I} G_i \ .$$

Now choose any $x \in U$. If $x \notin \text{dom } F$ then the desired inclusion follows trivially since $F(x) = \phi$. If $x \in \text{dom } F$ then let y be any point in $F(x)$; we have

$$(x,y) \in (U \times \mathbb{R}^m) \cap \text{graph } F \subset \bigcup_{i \in I} G_i \ ,$$

so for some $i \in I$, $(x,y) \in G_i$, and we know that both $F_i(x)$ and $F_i(x_0)$ are nonempty. Therefore,

$$y \in F_i(x) \subset F_i(x_0) + \lambda_i \|x - x_0\|B \subset F(x_0) + \lambda\|x - x_0\| B \ ,$$

since $F(x_0) = \bigcup_{i \in I} F_i(x_0)$. However, y was arbitrary in $F(x)$, so $F(x) \subset F(x_0) + \lambda\|x - x_0\|B$, and the proof is complete.

The essential tool in the proof just given was the theorem of Walkup and Wets; in [3] the author gave a different proof using Hoffman's theorem, but the argument needed there was somewhat longer.

An interesting consequence of this proposition can be developed by introducing the point-to-set distance defined by

$$d[x,A] := \inf\{\|x-a\| \mid a \in A \},$$

where by convention the infimum is $+\infty$ if A is empty.

COROLLARY: _Let_ F _be a polyhedral multifunction from_ $\mathbb{R}^n$ _to_ $\mathbb{R}^m$, _and let_ μ _be the modulus associated with_ F^{-1} _by Proposition 1. Then for each_ $y \in \mathbb{R}^m$ _there is some_ $\varepsilon > 0$ _such that if_ $x \in \mathbb{R}^n$ _with_ $d[y,F(x)] < \varepsilon$ _then_

$$d[x,F^{-1}(y)] \leq \mu d[y,F(x)] . \tag{5}$$

PROOF: By Proposition 1 there is some neighborhood of y on which F^{-1} is upper Lipschitzian at y with modulus μ. Choose ε so that the ball about y with radius ε is included in this neighborhood. Now if $x \in \mathbb{R}^n$ with $d[y,F(x)] < \varepsilon$ then since $F(x)$ is closed there is some $y_1 \in F(x)$ with $d[y,F(x)] = \|y-y_1\| < \varepsilon$. Therefore,

$$F^{-1}(y_1) \subset F^{-1}(y) + \mu\|y-y_1\|B ,$$

but since $x \in F^{-1}(y_1)$ we have

$$d[x,F^{-1}(y)] \leq \mu\|y-y_1\| = \mu d[y,F(x)],$$

as was to be shown.

We observe that this proof does not depend essentially upon polyhedrality, but rather upon the local upper Lipschitz continuity of F^{-1}, and therefore a version of corollary could be established also for non-polyhedral multifunctions whose inverses have this property. The quantity $d[y,F(x)]$ can be regarded as a kind of "residual" measuring the extent to which x does not satisfy the relation $y \in F(x)$.

As an illustration of why $d[y,F(x)]$ has to be sufficiently small in this result, consider the derivative of the function $f:\mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} \tfrac{1}{2}x^2 , & x \in [-1,1] \\ |x| , & x \notin [-1,1] . \end{cases}$$

One has

$$f'(x) = \begin{cases} 1 , & x > 1 \\ x , & x \in [-1,1] \\ -1, & x < 1 , \end{cases}$$

so that f' is a polyhedral multifunction (although it is actually single-valued), and therefore so is f'^{-1}. For $y = 0$, we see that if $d[0, f'(x)]$ is small then x is close to 0, and for such x it is true that

$$d[x, f'^{-1}(0)] = \|x\| \leq \mu d[0, f'(x)] = \mu \|x\| ,$$

in fact with $\mu = 1$. But for large x $d[0, f'(x)]$ is never larger than 1, whereas $d[x, f'^{-1}(0)] = \|x\|$ increases without bound, so that (5) does not hold for such x.

For a polyhedral _convex_ multifunction F, however, (5) holds for all $x \in \mathbb{R}^n$ and $y \in \text{im } F$. This can be seen directly by observing that

$$F^{-1}(y) = \pi_1 \{ \Gamma_F \cap \pi_2^{-1}(y) \} ,$$

and this is Lipschitzian (in the Hausdorff metric) on $\text{im } F$ by the Walkup-Wets theorem. If $y \in \text{im } F$ and $x \in \text{dom } F$, then by choosing $y_1 \in F(x)$ with $\|y - y_1\| = d[y, F(x)]$ one has for some μ,

$$x \in F^{-1}(y_1) \subset F^{-1}(y) + \mu \|y - y_1\| B ,$$

so that $d[x, F^{-1}(y)] \leq \mu d[y, F(x)]$; on the other hand, if $x \notin \text{dom } F$ then the inequality holds trivially because $d[y, F(x)] = +\infty$

Our first proposition showed that the images of points near a given point were near the image of that point (but perhaps not _vice versa_). Our second shows that any bounded subset of $\text{im } F$ comes from some bounded subset of $\text{dom } F$.

PROPOSITION 2: Let F be a polyhedral multifunction from $\mathbb{R}^n$ to $\mathbb{R}^m$. If K is any bounded subset of $\text{im } F$ then there is a bounded set $M \subset \mathbb{R}^n$ such that $F(M) \supset K$.

PROOF: The image of F is the union of finitely many sets of the form $\pi_2(G_i)$, where G_i is a component of the graph. Since the linear image of a polyhedral convex set is closed, $\text{im } F$ is closed; hence $\text{cl } K$ is also a bounded subset of $\text{im } F$, and so we may assume with no loss of generality that K is compact. Choose any point $y_0 \in K$, and apply the lemma to F^{-1} in order to produce a neighborhood V of y_0 such that $V \cap \text{im } F = \cup_{i=1}^k [V \cap \pi_2(G_i)]$, where the G_i are components of the graph of F with $y_0 \in \cap_{i=1}^k \pi_2(G_i)$. Let W be any bounded polyhedral convex neighborhood of y_0 contained in V, and define $H_i := W \cap \pi_2(G_i)$ for $i = 1, \ldots, k$. The sets H_i are compact convex polyhedra, and thus each is the convex hull of its (finitely many) extreme points. Therefore, for each i the convex function $f_i(y) := \inf\{\|x\| \mid (x,y) \in G_i\}$ attains a maximum on H_i (at one of the extreme points), say μ_i. Let $\mu := \max_{i=1}^k \mu_i$. If $y \in W \cap \text{im } F$, one has for some i, $y \in H_i$; thus, there exists x with $(x,y) \in G_i$

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(so that $y \in F(x)$) and $\|x\| \leq f_i(y) \leq \mu_i \leq \mu$. Therefore $F(\mu B) \supset W \cap \operatorname{im} F$. Now, a relative neighborhood of the form $W \cap \operatorname{im} F$ can be constructed about each point of K, and by compactness a finite number of these will cover K. As each of them is contained in the image under F of some ball, we have the result.

By recalling that the linear complementarity problem can be written in the form (1) (with $C = \mathbb{R}^n_+$) we see that the results of Propositions 1 and 2 can be applied immediately to such problems (or to other linear generalized equations with polyhedral sets C). The solutions of such problems therefore obey the local Lipschitz condition described in Proposition 1, and by Proposition 2 if the "constant term" denoted by $\underline{a}$ in (1) is allowed to vary in a bounded set, then for each such $\underline{a}$ either the solution set is empty or its distance from the origin obeys a uniform bound (although, of course, the solution set itself may be unbounded).

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3. Applications to parametric programming. In this final section, we apply Propositions 1 and 2 to the problem of parametric optimization. We shall consider the function $Af: \mathbb{R}^m \to [-\infty, +\infty]$ given by

$$Af(z) := \inf_x \{f(x,z) \mid x \in A^{-1}(z)\}, \tag{6}$$

where $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and where A is a polyhedral multifunction from $\mathbb{R}^m$ to $\mathbb{R}^n$. The function Af expresses the optimal value in (6) as a function of the parameters z; the notation Af is suggested by that of [6]. We define the effective domain of Af by $\text{dom } Af = \{z \mid Af(z) < +\infty\}$. We shall prove two general continuity results for Af, and then illustrate one of these results in the simple case of convex quadratic programming. For ease of writing, we shall norm $\mathbb{R}^n \times \mathbb{R}^m$ by the sum of the norms on $\mathbb{R}^n$ and $\mathbb{R}^m$.

PROPOSITION 3: Let $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and let A be a polyhedral multifunction from $\mathbb{R}^n$ to $\mathbb{R}^m$. Suppose that f is Lipschitzian on bounded subsets of $\mathbb{R}^n \times \mathbb{R}^m$. If z_0 is a point of $\text{dom } Af$ such that $A^{-1}(z_0)$ is bounded, then for each z near z_0 one has

$$Af(z) \geq Af(z_0) - L\|z - z_0\| \tag{7}$$

for some L which depends only on $\|z_0\|$ and the bound for $A^{-1}(z_0)$.

PROOF: Let two numbers α and β be specified, and let f be Lipschitzian with modulus λ on $\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m \mid \|x\| \leq \beta+1, \|z\| \leq \alpha\}$. Let $\mu > 0$ be a local upper Lipschitz modulus for A^{-1} (Proposition 1), and define $L := \lambda(1+\mu)$. Choose any $z_0 \in \alpha B$ with $A^{-1}(z_0) \subset \beta B$. Let N be a neighborhood of z_0 such that $N \subset z_0 + \mu^{-1} B$ and for each $z \in N$, $A^{-1}(z) \subset A^{-1}(z_0) + \mu\|z - z_0\| B$, and choose $z \in N$. If $z \notin \text{dom } Af$ then (7) is certainly true. If $z \in \text{dom } Af$, then $z \in \text{dom } A^{-1}$. Choose $\varepsilon > 0$ and find $x \in A^{-1}(z)$ with $|f(x,z) - Af(z)| \leq \varepsilon$. Observe that since $A^{-1}(z) \subset A^{-1}(z_0) + \mu\|z - z_0\|$, one has (i) $\|x\| \leq \beta + \mu\mu^{-1} = \beta+1$, and (ii) for some $x_0 \in A^{-1}(z_0)$, $\|x - x_0\| \leq \mu\|z - z_0\|$. Then

$$Af(z_0) \leq f(x_0, z_0) = [f(x_0, z_0) - f(x,z)] + f(x,z)$$
$$\leq \lambda(\|x - x_0\| + \|z - z_0\|) + [Af(z) + \varepsilon]$$
$$\leq Af(z) + L\|z - z_0\| + \varepsilon.$$

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Letting $\varepsilon \downarrow 0$, we obtain (7), which finishes the proof.

For this result we assumed the (strong) condition that the feasible set $A^{-1}(z_0)$ was bounded. We can remove this hypothesis if we assume that the set of optimal solutions of the minimization problem is a polyhedral multifunction.

PROPOSITION 4: Let $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and let A be a multifunction from $\mathbb{R}^n$ to $\mathbb{R}^m$. Suppose that f is Lipschitzian on bounded subsets of $\mathbb{R}^n \times \mathbb{R}^m$ and that the multifunction $P: \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$P(z) := \{x \in A^{-1}(z) \mid f(x,z) = Af(z)\},$$

is polyhedral. Then for each bounded $Q \subset \mathbb{R}^m$ there is a constant L such that if $z_0 \in Q \cap \operatorname{dom} P$ then for each $z \in \operatorname{dom} P$ near z_0 ,

$$|Af(z) - Af(z_0)| \leq L\|z-z_0\|. \tag{8}$$

In this case Af is Lipschitzian on each bounded convex subset of dom P.

PROOF: Let ν be the local upper Lipschitz constant for P (Proposition 1). Select some bounded set $Q \subset \mathbb{R}^m$; let $Q \subset \alpha B$ and let β be large enough that $P^{-1}(\beta B) \supset [(\alpha+1)B] \cap \operatorname{dom} P$ (Proposition 2). Let f be Lipschitzian on $\{(x,z) \mid \|x\| \leq \beta+\nu, \|z\| \leq \alpha+1\}$ with modulus λ , and define $L := \lambda(\nu+1)$. Now choose $z_0 \in Q \cap \operatorname{dom} P$; let N be a neighborhood of z_0 contained in $z_0 + B$ and such that if $z \in N$ then $P(z) \subset P(z_0) + \nu\|z-z_0\|B$. Choose any $z \in N \cap \operatorname{dom} P$; then since $z \in (\alpha+1)B \cap \operatorname{dom} P$ there is some $x \in P(z)$ with $\|x\| \leq \beta$. There is then some $x_0 \in P(z_0)$ with $\|x-x_0\| \leq \nu\|z-z_0\| \leq \nu$ (so that $\|x_0\| \leq \beta+\nu$), and we have

$$|Af(z) - Af(z_0)| = |f(x,z) - f(x_0,z_0)| \leq \lambda(\|x-x_0\| + \|z-z_0\|)$$
$$\leq \lambda(\nu+1)\|z-z_0\| = L\|z-z_0\|.$$

The proof that Af is Lipschitzian on bounded convex subsets of dom P is a routine exercise once (8) is established, and we omit it.

To show how Proposition 4 may be applied, we return to the quadratic programming problem (3). If we assume that H is positive semidefinite (i.e., that the objective function in (3) is convex), then the conditions (2) are necessary and sufficient for optimality. Writing $z = (c,b)$, defining F by (4) and f by

$$f(y,z) := \tfrac{1}{2}\langle y, Hy \rangle + \langle c, y \rangle,$$

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with

$$A(y) := \begin{cases} \mathbb{R}^n \times (Dy - Q^*), & \text{if } y \in P \\ \emptyset, & \text{if } y \notin P, \end{cases}$$

we find that $Af(z)$ is the optimal value of (3), so that

$$P(z) = \{y \in A^{-1}(z) \mid f(y,z) = Af(z)\}$$
$$= \{y \mid (y,u) \in F^{-1}(z) \text{ for some } u\}$$
$$= (\pi_1 F^{-1})(z) .$$

Thus P is polyhedral, and f is certainly Lipschitzian on bounded sets, so Proposition 4 tells us that the optimal value $Af(c,b)$ is Lipschitzian on bounded subsets of dom F^{-1}. In this case dom F^{-1} is a closed convex set, since F is **maximal** monotone and polyhedral; thus the subsets need not be assumed **convex**.

For some other applications of polyhedral multifunctions to stability analysis, see [4] and [5].

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A multifunction is polyhedral if its graph is the union of finitely many polyhedral convex sets. This paper points out some fairly strong continuity properties that such multifunctions satisfy, and it shows how these may be applied to such areas as linear complementarity and parametric programming.

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